كلية الحاسبات والذكاء الإصطناعي

BS111
Probability and Statistics

Lecture 09

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Chapter 3: Prob. Distributions

- Discrete Uniform Distribution.
- Binomial Distribution.
- Continuous Uniform Distribution.
- Normal Distribution.

Discrete Uniform Dist. (1/6)

Definition:

A random variable X has a discrete uniform distribution if each of the n values in its range, $x_1, x_2, \ldots, x_n$, has equal probability. Then

$$f(x_i) = \frac{1}{n}$$

Discrete Uniform Dist. (2/6)

Example1 (1/2):

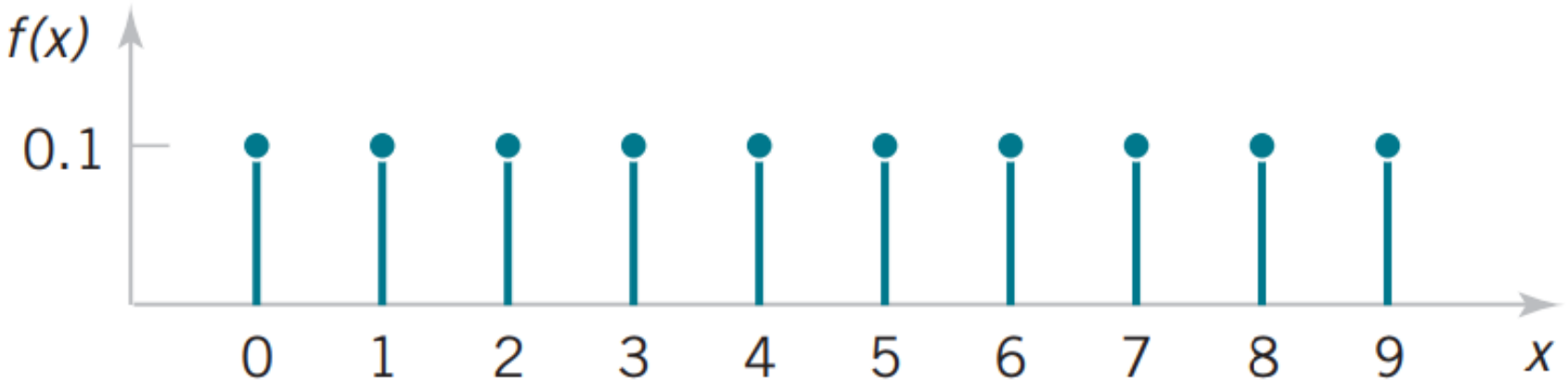
The first digit of a part's serial number is equally likely to be any one of the digits 0 through 9. If one part is selected randomly from a large batch and X is the first digit of the serial number, X has a discrete uniform distribution with probability 0.1 for each value in $R = \{0, 1, 2, \dots, 9\}$. That is,

$$f(x) = \frac{1}{10} = 0.1$$

Discrete Uniform Dist. (2/6)

Example1 (2/2):

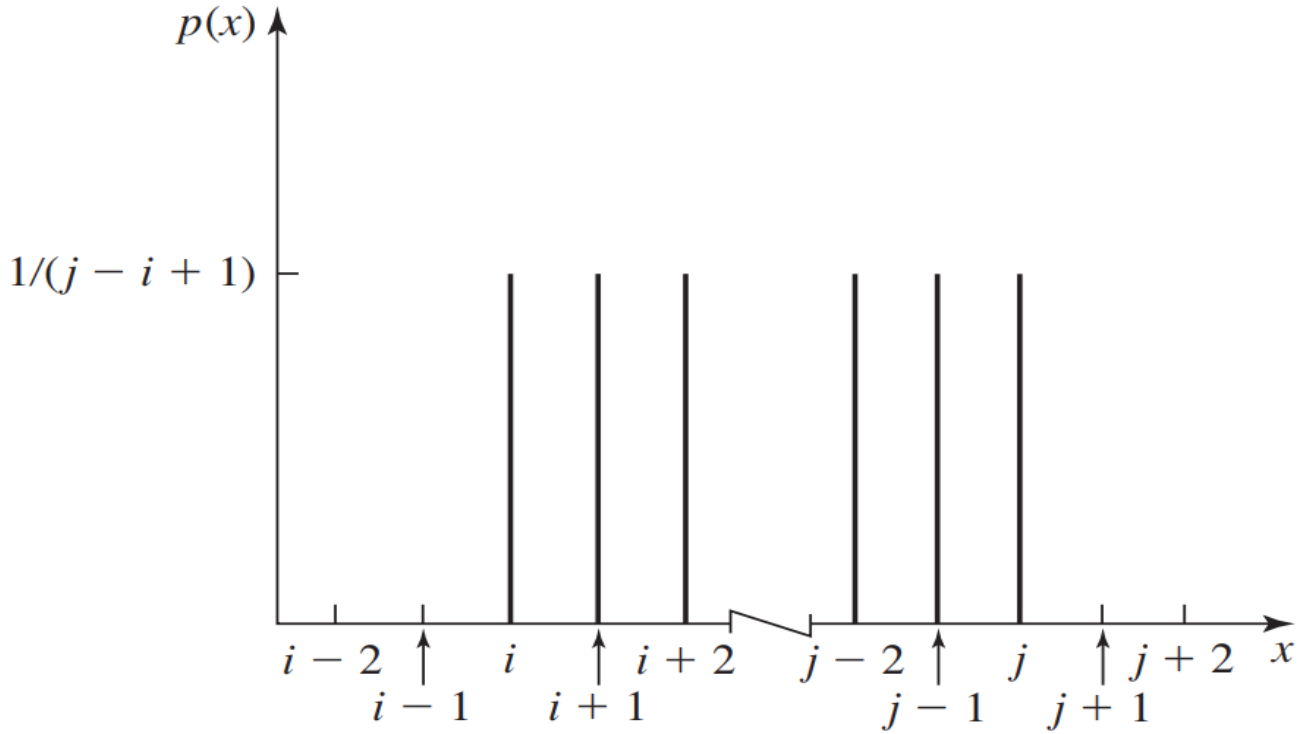
Probability mass function for a discrete uniform random variable



Discrete Uniform Dist. (3/6)

Discrete Uniform $DU(i, j)$:

Suppose that the range of the discrete random variable X equals the consecutive integers $i, i + 1, \ldots, j$, for $i \leq j$



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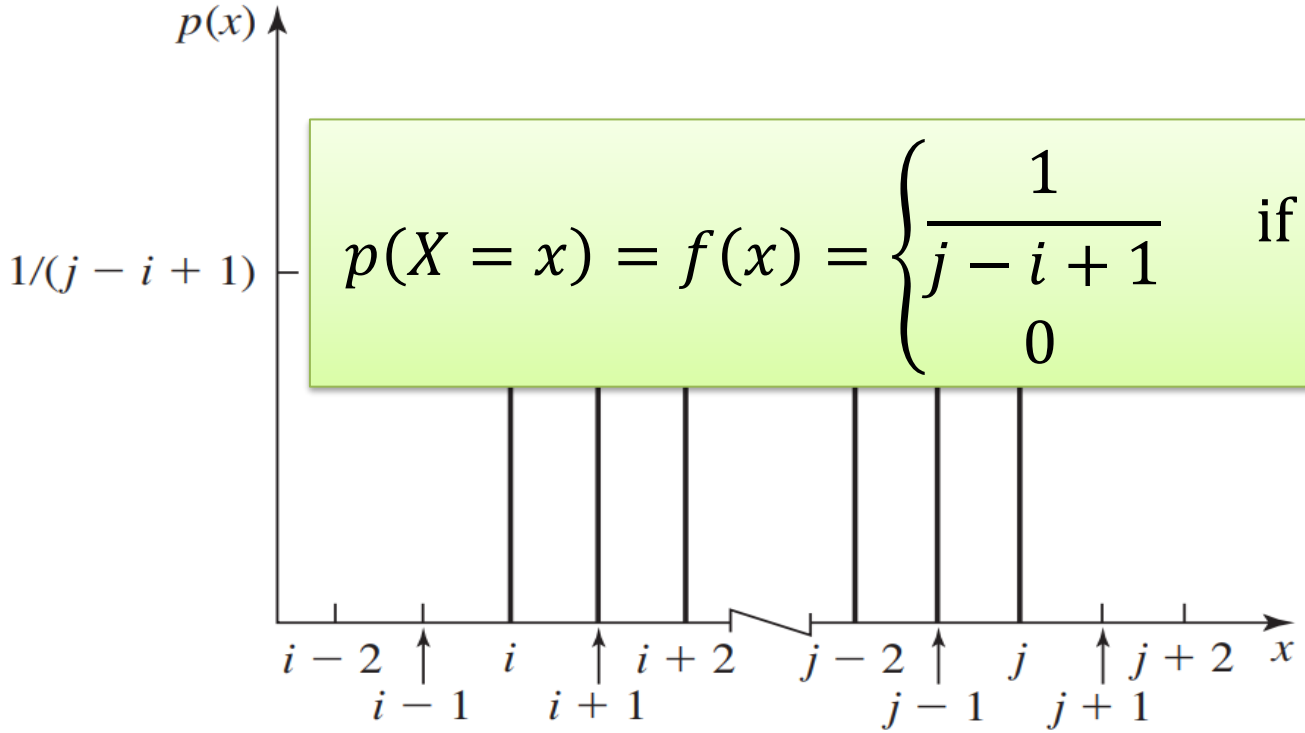
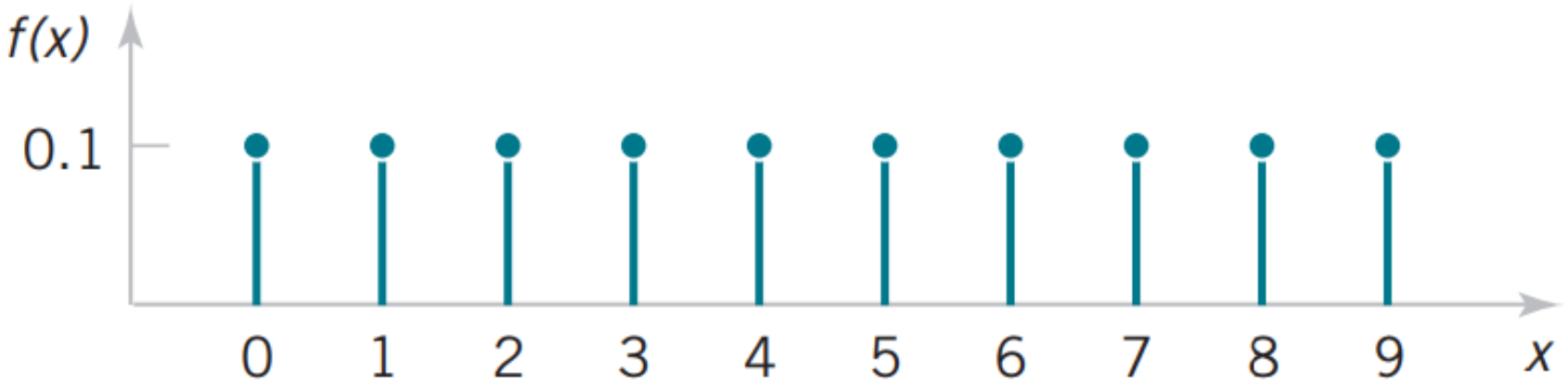


$$p(X = x) = f(x) = \begin{cases} \dfrac{1}{j - i + 1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

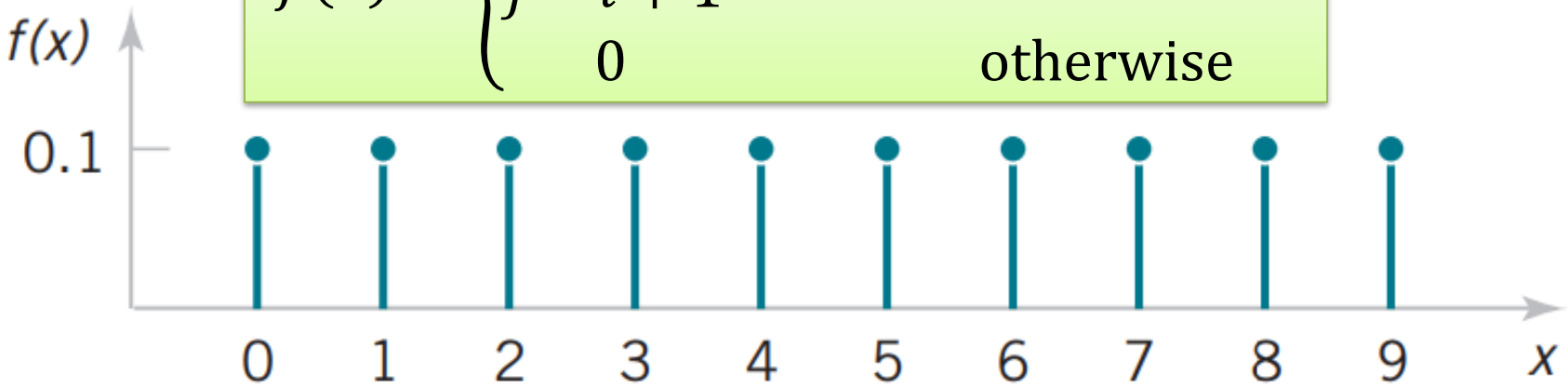


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{j-i+1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

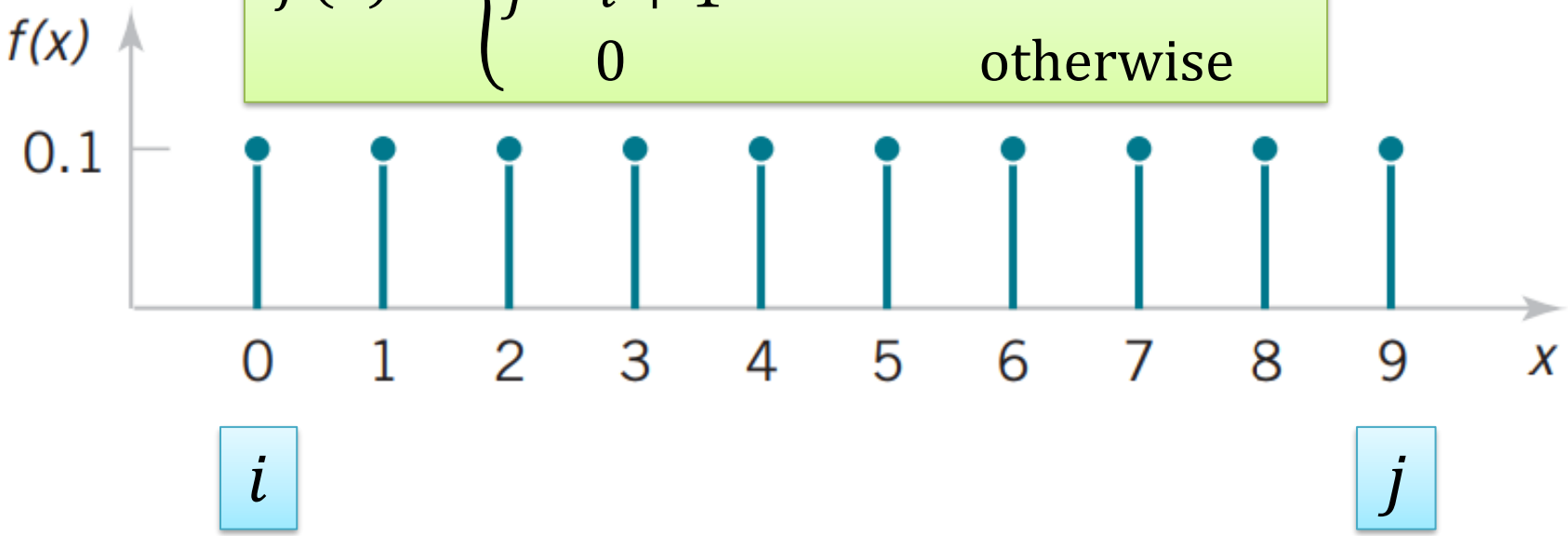


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{j-i+1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

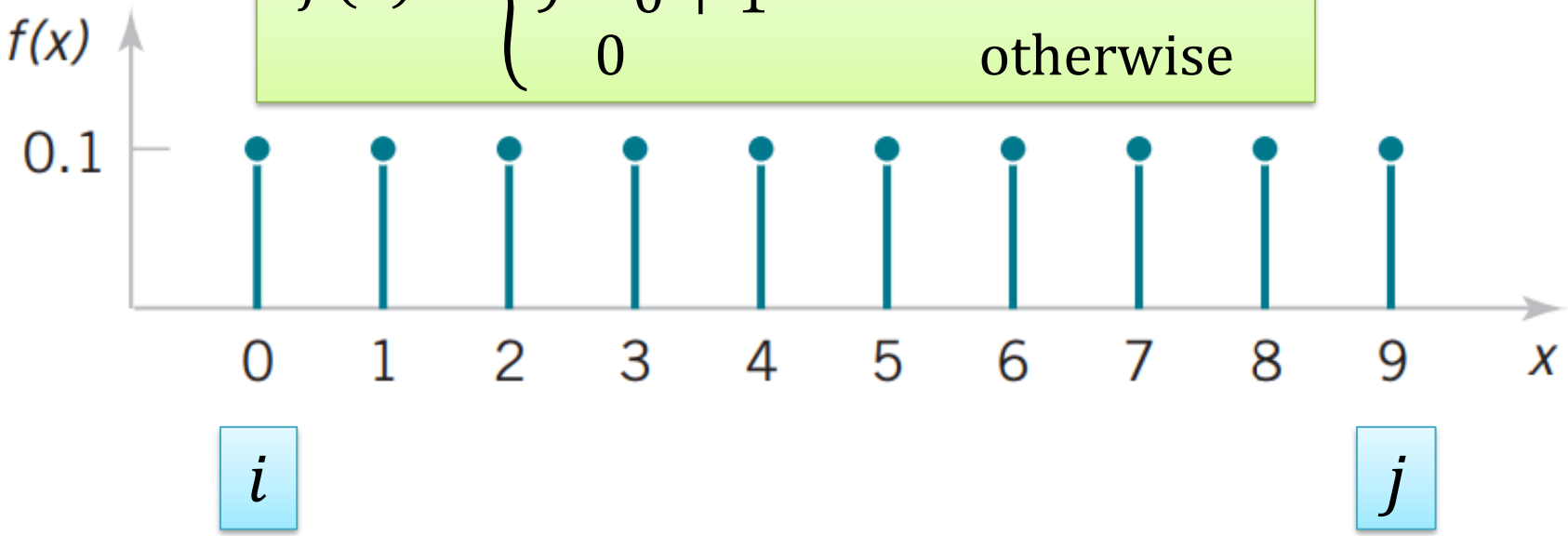


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{9-0+1} & \text{if } x \in \{0, 1, \ldots, 9\} \\ 0 & \text{otherwise} \end{cases}$$

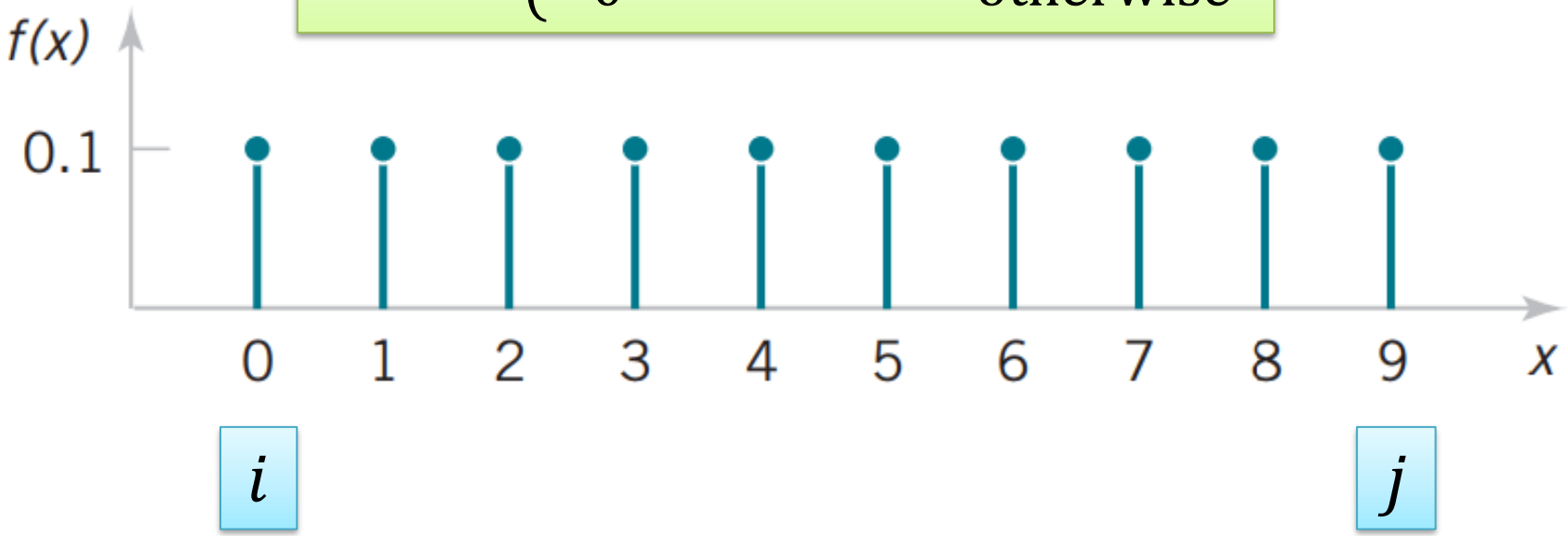


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} 1/10 & \text{if } x \in \{0, 1, \ldots, 9\} \\ 0 & \text{otherwise} \end{cases}$$



Discrete Uniform Dist. (5/6)

Mean and Variance:

Suppose that the range of the discrete random variable X equals the consecutive integers $i, i+1, \ldots, j$, for $i \leq j$

$$\mu = E(X) = \frac{j+i}{2}$$

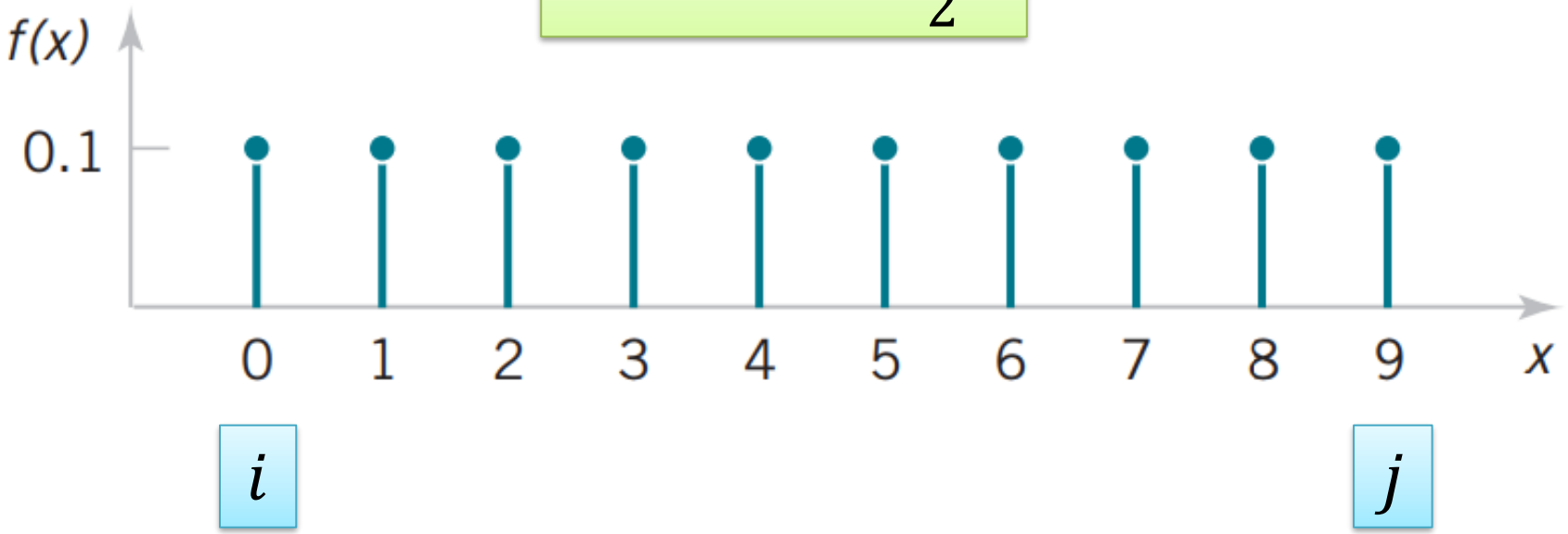
$$\sigma^2 = V(X) = \frac{(j-i+1)^2 - 1}{12}$$

Discrete Uniform Dist. (6/6)

Example2 (1/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\mu = E(X) = \frac{j+i}{2}$$

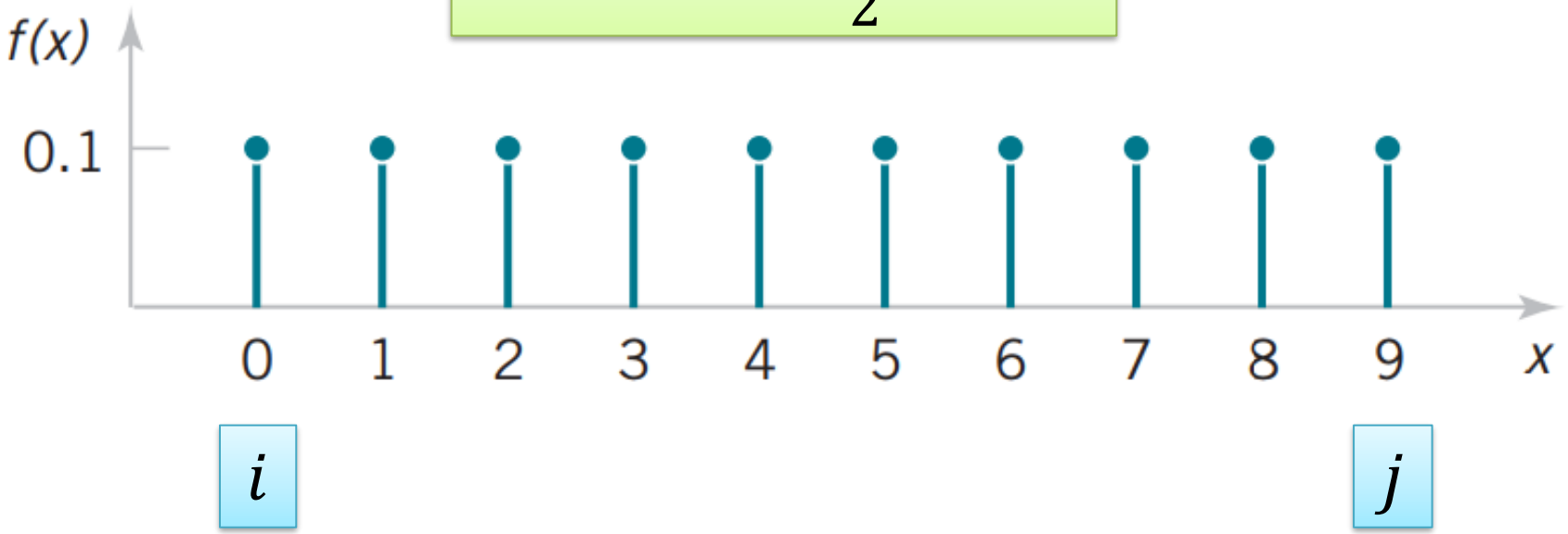


Discrete Uniform Dist. (6/6)

Example2 (2/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\mu = E(X) = \frac{9+0}{2} = 4.5$$

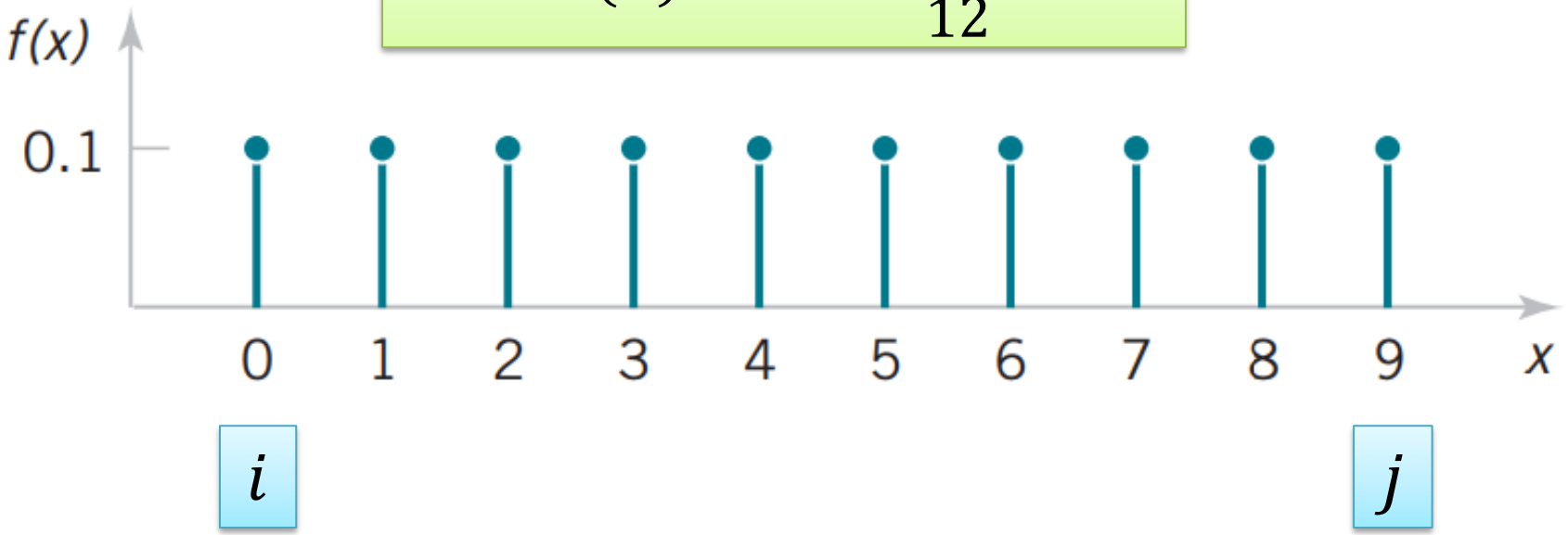


Discrete Uniform Dist. (6/6)

Example2 (3/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\sigma^2 = V(X) = \frac{(j-i+1)^2 - 1}{12}$$

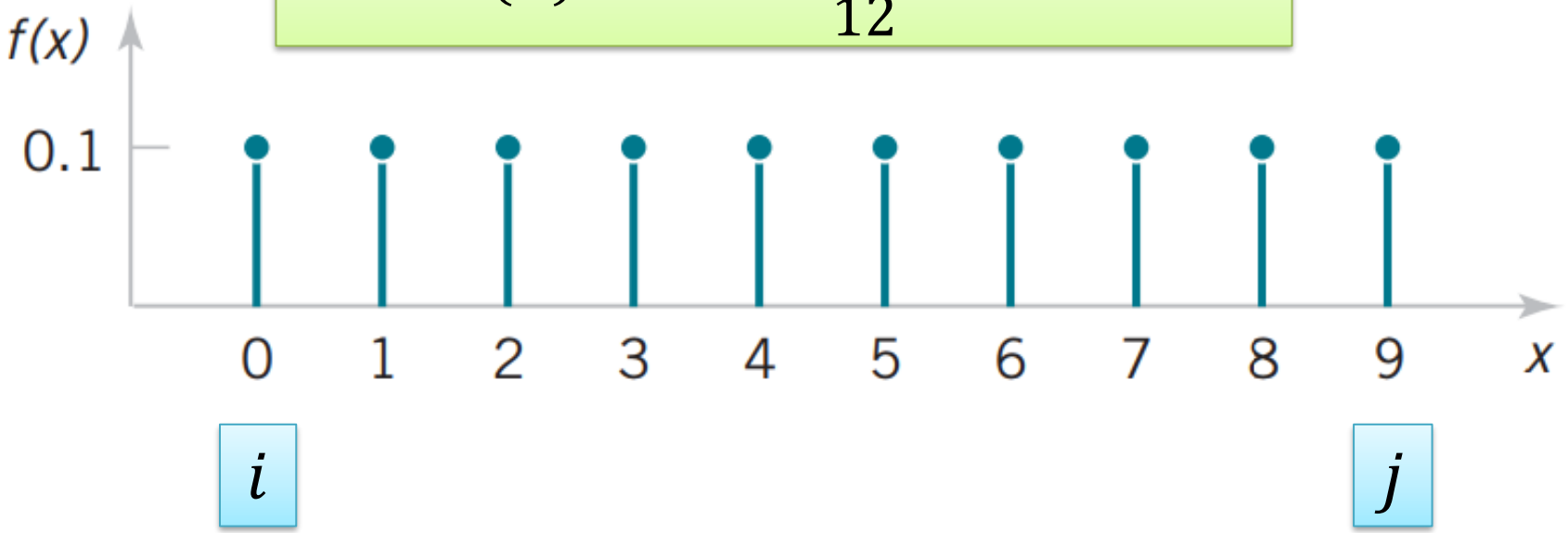


Discrete Uniform Dist. (6/6)

Example2 (4/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\sigma^2 = V(X) = \frac{(9-0+1)^2-1}{12} = 8.25$$



Binomial Distribution (1/10)

The Bernoulli Process:

1. The experiment consists of ***repeated*** trials and each trial is called a ***Bernoulli trial***.
2. Each trial results in an outcome that may be classified as a ***success*** or a ***failure***.
3. The probability of success, denoted by $\boldsymbol{p}$, remains constant from trial to trial.
4. The repeated trials are ***independent***.

Binomial Distribution (2/10)

The Bernoulli Process: (Examples)

1. Flip a coin 10 times. Let X = number of heads obtained.
2. A multiple-choice test contains 10 questions, each with four choices, and you guess at each question. Let X = the number of questions answered correctly.
3. In the next 20 births at a hospital, let X = the number of female births.

Binomial Distribution (3/10)

Binomial Distribution $b(n, p)$:

A random experiment consists of n Bernoulli trials. The random variable X that equals the number of trials that result in a success is a *binomial random variable* with parameters $0 < p < 1$ and finite $n = 1, 2, ...$

The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

Binomial Distribution (4/10)

Example 1 (1/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

Binomial Distribution (4/10)

Example 1 (2/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/10)

Example 1 (3/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/10)

Example 1 (4/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/10)

Example 1 (5/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(3) = \binom{10}{3} (0.5)^3 (0.5)^{10-3}$$

$$= 120\ (0.5)^3 (0.5)^7$$

$$= 0.1171875$$

Binomial Distribution (5/10)

Example 2 (1/5):

Flip a coin 10 times, what is the probability that:

a) No head occurs.

b) At least 2 heads occurs.

Binomial Distribution (5/10)

Example 2 (2/5):

Flip a coin 10 times, what is the probability that:

a) No head occurs.

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

Binomial Distribution (5/10)

Example 2 (3/5):

Flip a coin 10 times, what is the probability that:

a) No head occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$f(0) = \binom{10}{0} (0.5)^0 (0.5)^{10-0}$$

Binomial Distribution (5/10)

Example 2 (4/5):

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

Binomial Distribution (5/10)

Example 2 (4/5):

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$P(X \geq 2)$

Binomial Distribution (5/10)

Example 2 (4/5):

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$P(X \geq 2) = f(2) + f(3) + \cdots + f(10)$$

Binomial Distribution (5/10)

Example 2 (4/5):

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$P(X \geq 2) = \cancel{f(2) + f(3) + \cdots + f(10)}$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [f(1) + f(0)]$$

Binomial Distribution (5/10)

Example 2 (5/5):

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(0) = \binom{10}{0}(0.5)^0(0.5)^{10-0}$$

$$f(1) = \binom{10}{1}(0.5)^1(0.5)^{10-1}$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [f(1) + f(0)]$$

Binomial Distribution (6/10)

Mean and Variance $b(n, p)$:

If X is a binomial random variable with parameters p and n

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/10)

Example 3 (1/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

Binomial Distribution (7/10)

Example 3 (2/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (7/10)

Example 3 (3/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/10)

Example 3 (4/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$
$$= 10 \times 0.5 = 5$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/10)

Example 3 (5/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$
$$= 10 \times 0.5 = 5$$

$$\sigma^2 = V(X) = np(1-p)$$
$$= 10 \times 0.5 \times 0.5 = 2.5$$

Binomial Distribution (8/10)

Example 4 (1/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

Binomial Distribution (8/10)

Example 4 (2/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

Binomial Distribution (8/10)

Example 4 (3/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

Binomial Distribution (8/10)

Example 4 (3/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \ldots, 15$

Binomial Distribution (8/10)

Example 4 (4/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

(a) exactly 5 survive

$$f(5) = \binom{15}{5} (0.4)^5 (0.6)^{15-5}$$

$$= 0.1859$$

Binomial Distribution (8/10)

Example 4 (5/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

(b) at least 3 survive

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

$$P(X \geq 3) = 1 - P(X < 3)$$
$$= 1 - [f(2) + f(1) + f(0)]$$

Binomial Distribution (9/10)

Example 5 (1/4):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

Binomial Distribution (9/10)

Example 5 (2/4):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

$$\mu = E(X) = np$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (9/10)

Example 5 (3/4):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\mu = E(X) = np$$
$$= 15 \times 0.4 = 6$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (9/10)

Example 5 (4/4):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

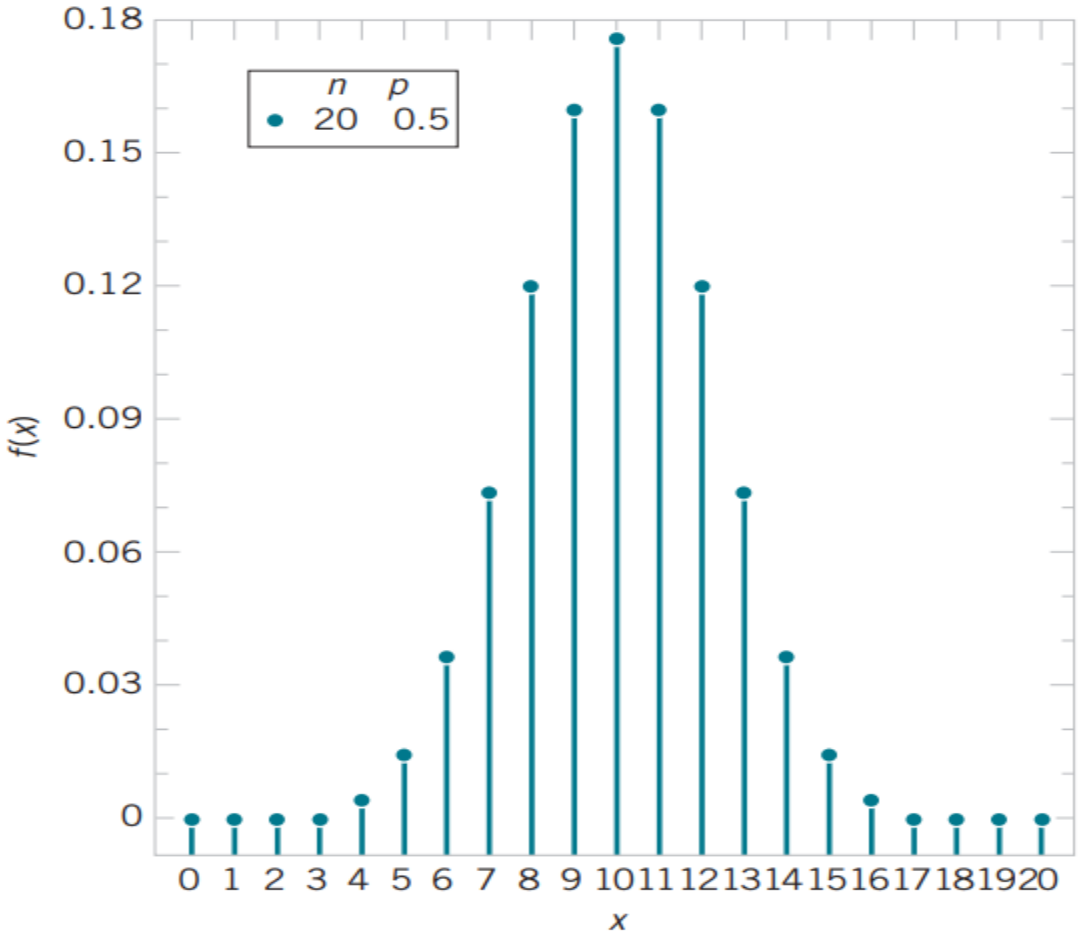
- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\mu = E(X) = np$$
$$= 15 \times 0.4 = 6$$

$$\sigma^2 = V(X) = np(1-p)$$
$$= 15 \times 0.4 \times 0.6$$
$$= 3.6$$

Binomial Distribution (10/10)

Binomial distribution for selected values of $n = 20$ and $p = 0.5$



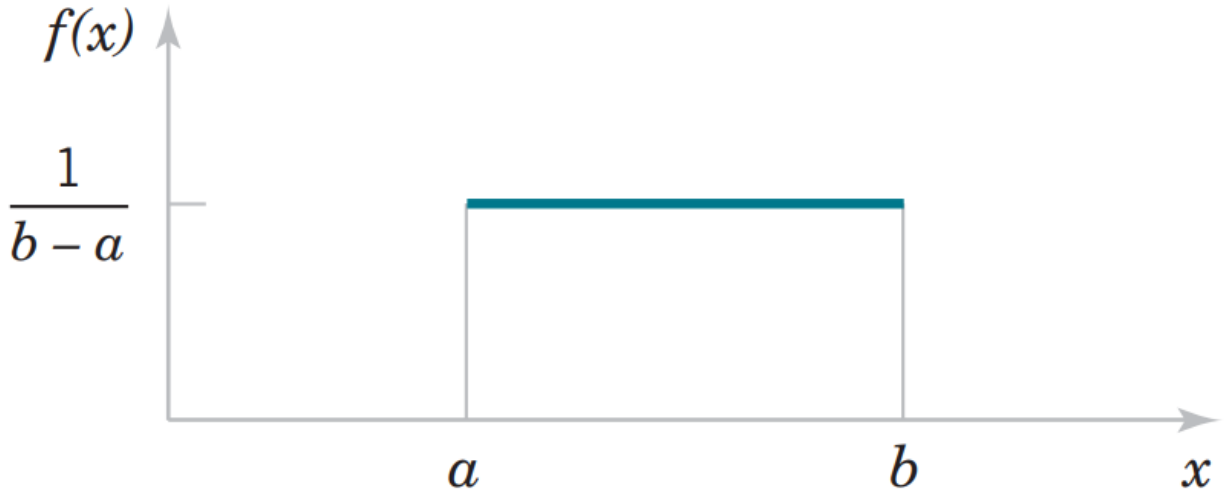
Continuous Uniform Dist. (1/2)

Continuous Uniform Distribution

A continuous random variable X with probability density function

$$f(x) = 1/(b-a), \quad a \le x \le b$$

is a **continuous uniform random variable**.



Continuous Uniform Dist. (2/2)

Mean and Variance

If X is a continuous uniform random variable over $a \leq x \leq b$,

$$\mu = E(X) = \frac{a+b}{2}$$

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

Normal Distribution (1/2)

Normal Distribution

A random variable X with probability density function

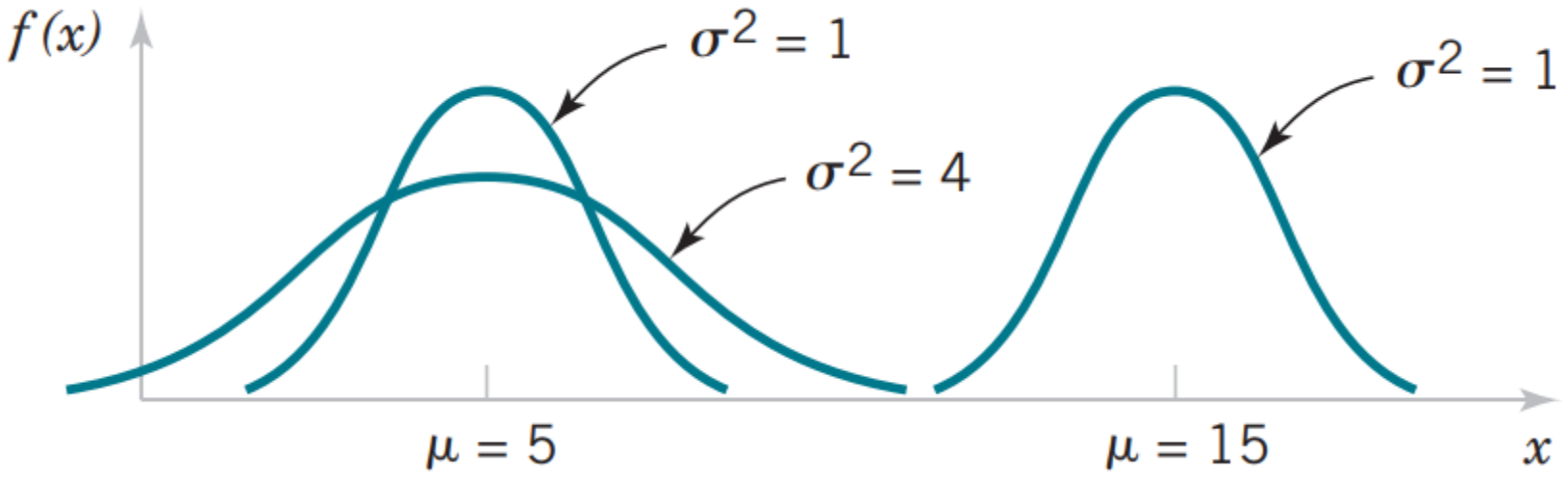
$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

is a **normal random variable** with parameters μ where $-\infty < \mu < \infty$ and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution.

Normal Distribution (2/2)



Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #9: https://www.youtube.com/watch?v=bRK10hfwtoY&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=11

Thank You

Dr. Ahmed Hagag
ahagag@fci.bu.edu.eg